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Property M and the fixed point property for nonexpansive mappings

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1. PRELIMINARIES

Let N_a be the set of all absolute normalized norms $\|\cdot\|$ on $\mathbb{C}^2$, that is, $\|(z,w)\| = \|(|z|,|w|)\|$ and $\|(1,0)\| = \|(0,1)\| = 1$, and Ψ the family of the convex (continuous) functions on the unit interval $[0,1]$ with

(1) $\quad \psi(0) = \psi(1) = 1$ and $\max\{1-t,t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1)$

By (Bonsall and Duncan [3], see also [16]), If, for some $\|\cdot\| \in N_a$, a convex function ψ is defined by

$$\psi(t) = \|(1-t,t)\| \quad (0 \leq t \leq 1), \tag{1}$$

then the function ψ is convex (continuous) and satisfies (1) and conversely, if, for any $\psi \in \Psi$, a norm $\|\cdot\|$ is define by

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0), \end{cases} \tag{2}$$

then $\|\cdot\|_\psi$ is an absolute normalized norm on $\mathbb{C}^2$ and satisfies (4). The ℓ_p-norms $\|\cdot\|_p$ are typical such examples and for any $\|\cdot\| \in N_a$ we have

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1 \tag{3}$$

([3]). By (4) the convex functions corresponding to the ℓ_p-norms are given by

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t,t\} & \text{if } p = \infty. \end{cases} \tag{4}$$

Let X and Y be Banach spaces and let $\psi \in \Psi$. The ψ-direct sum $X \oplus_\psi Y$ of X and Y is the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi, \tag{5}$$

where the $\|(\cdot,\cdot)\|_\psi$ term in the right hand side is an element of N_a with the corresponding convex function ψ. The following monotone properties were proved.

Proposition 1 ([3]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x,y),(z,w) \in X \oplus_\psi Y$. The following hold:

(i) $\|x\| \leq \|z\|$ and $\|y\| \leq \|w\|$ implies $\|(x,\ y)\|_\psi \leq \|(z,\ w)\|_\psi$.

(ii) $\|x\| < \|z\|$ and $\|y\| < \|w\|$ implies, $\|(x,\ y)\|_\psi < \|(z,\ w)\|_\psi$.

Proposition 2 ([17]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x,y),(z,w) \in X \oplus_\psi Y$. The following hold:

(i) $\|x\| \leq \|z\|$ and $\|y\| < \|w\|$, or, $\|x\| < \|z\|$ and $\|y\| \leq \|w\|$ implies $\|(x,\ y)\|_\psi < \|(z,\ w)\|_\psi$.

(ii) For $t \in (0,1)$, $\psi(t) > \psi_\infty(t)$ holds.

Proposition 3 ([10]). Let X, Y Banach spaces and $\psi \in \Psi$. And let $(x,y),(z,w) \in X \oplus_\psi Y$. The following hold:

(i) Let $\|x\| < \|z\|$ and $\|y\| = \|w\|$. $\|(z,w)\|_\psi = \|(x,y)\|_\psi$ if and only if $\|(z,w)\|_\psi = \|w\|$.

(ii) Let $\|x\| = \|z\|$ and $\|y\| < \|w\|$. $\|(z,w)\|_\psi = \|(x,y)\|_\psi$ if and only if $\|(z,w)\|_\psi = \|z\|$.

Let Y be a Banach space and P a projection on Y. P is called an L(M)-projection if $x = \|Px\| + \|(id_Y - P)x\|(\max\{\|x\|, \|(id_Y - P)x\|\})$ for all $x \in Y$, respectively. Let $X^\perp \subset X^{***}$ be annihilater of X, i.e. $X^\perp = \{w \in X^{***} : w(x) = 0, \forall x \in X\}$. A closed subspace $X \subset Y$ is called an L(M)-summand on Y if X is the range of an L(M)-projection on Y. It is said that X is an L(M)-embedded Banach space if there exists a closed subspace $X_s \subset X^{**}$ such that $X^{**} = X \oplus_1 X_s (X^{***} = X^* \oplus_1 X^\perp)$.(Cf. [8].) For some $\psi \in \Psi$, we shall introduce $\psi(\psi^*)$-embedded Banach space if there exists a closed subspace $X_s \subset X^{**}$ such that $X^{**} = X \oplus_\psi X_s (X^{***} = X^* \oplus_\psi X^\perp)$.

Let $\{x_n\}$ is a sequence of a Banach space X, it is said that $\{x_n\}$ spans an asymptotically isometric copy of ℓ_1 if there exists a nonincreasing sequence $\{\delta_n\} \subset [0,1)$ tending to 0 such that

$$\sum (1-\delta_n)|\alpha_n| \leq \|\sum \alpha_n x_n\| \leq \sum |\alpha_n|$$

for every $\{\alpha_n\} \in \ell_1$. In this case we will denote $x_n \sim_{(asy)} \ell_1$.

The abstract measure topology (τ_μ) is defined by considering the class of convergent sequences.(Cf. [8].) Namely, if $\{x_n\}$ is a sequence in a Banach space X, we say that $\{x_n\}$ tends to 0 in the abstract measure topology $(\tau_\mu - \lim_n x_n = 0)$ if $\{x_n\}$ is norm bounded and every subsequence $\{x_{n_k}\}$ contains a subsequence $\{x_{n_{k_\ell}}\}$ such that $x_{n_{k_\ell}} \sim_{(asy)} \ell_1$ or $x_{n_{k_\ell}} \to 0$. A sequence $\{x_n\}$ tends to x in τ_μ if

($\tau_\mu - \lim_n(x_n - x) = 0$) and a subset $A \subset X$ is τ_μ-closed if it is τ_μ -sequentially closed.

Pfizner[13] proved the following theorem.

Theorem A.([13]) *Let X be an L-embedded Banach space ($X^{**} = X \oplus_1 X_s$). Let P be the natural projection on X^{**} with range X, and consider $C \subset X$ which is closed, bounded and convex. Then the following two assertions are equivalent:*

(i) $P(cl_{\sigma(X^{**},X^*)}C) = C$.
(ii) C is closed for the abstract measure topology.

By Thoerem A, Japón Pineda[14] proved the following theorems.

Theorem B.([14]) *Let X be an L-embedded Banach space. If C is a convex, bounded, closed for the abstract measure topology, subset of X which is diametral, then C is weakly compact.*

Theorem C.([14]) *Let X be the dual of an M-embedded space E. Then the abstract measure topology τ_μ is finer than the $\sigma(X, E)$ topology on bounded subsets of X.*

Theorem D.([14]) *Let X be the dual of an M-embedded space E. Then the following are equivalent:*
(i) X has the $\sigma(X, X^*)$-FPP.
(ii) X has the $\sigma(X, E)$-FPP.

By introducing the concept of ψ^*-embedded Banach space E, we obtained a generalization of Theorem C.

Theorem 1. *Let X be the dual of an ψ^*-embedded Banach space E with $\psi > \psi_\infty$. If C is a $\sigma(X, E)$-compact subset of X which is diametral, then C is weakly compact.*

By the Theorem 1, we obtain a generaization of Theorem D.

Theorem 2. *Let X be the dual of an ψ^*-embedded Banach space E with $\psi > \psi_\infty$. The following are equivalent:*
(i) X has the $\sigma(X, X^*)$-FPP.
(ii) X has the $\sigma(X, E)$-FPP.

Banach space X has Schur propety if every weakly convergent sequence of X converges strongly.

Using a Dominguez's generalization of the Garcia-Falset coefficient $R(X)$, $M(X)$([6]) is defined by

$$M(X) = \sup\left\{\frac{1+a}{R(a,X)} : a \geq 0\right\},$$

where

$$R(a,X) = \sup\left\{\liminf_{n\to\infty}\|x_n + x\|\right\},$$

where the supremum is taken over all $a > 0$, x with $\|x\| \leq a$ and weakly null sequences $\{x_n\}$ of the unit ball of X such that its double limit of $\{\|x_n - x_m\|\}_{n,m}$ exists and $\lim_n \lim_m \|x_n - x_m\| \leq 1$.

The following theorem is known.

Theorem E (cf. [1]). *Let X be a Banch space. If $M(X) > 1$, then X has weak fiexd point property.*

The following two lemma have important roles for proving Theorem5.

Lemma 1. Let $\{x_n^{(k)}\}, \{y_n^{(k)}\}$ of a Banach space X be nonzero double sequences with $\lim_{n\to\infty}\|x_n^{(k)}\| > 0$, $\lim_{n\to\infty}\|y_n^{(k)}\| > 0$ for each k . The following are equivalent.

(i) $\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\|x_n^{(k)} + y_n^{(k)}\| = \lim\limits_{k\to\infty}\lim\limits_{n\to\infty}(\|x_n^{(k)}\| + \|y_n^{(k)}\|)$.

(ii) $\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\left\|\dfrac{x_n^{(k)}}{\|x_n^{(k)}\|} + \dfrac{y_n^{(k)}}{\|y_n^{(k)}\|}\right\| = 2.$

Lemma 2. Let $\{x_n^{(k)}\}, \{y_n^{(k)}\}$ of a Banach space X be nonzero double sequences with $\lim_{n\to\infty}\|x_n^{(k)}\| > 0$, $\lim_{n\to\infty}\|y_n^{(k)}\| > 0$ for each k . The following are equivalent.

(i) $\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\|x_n^{(k)} + y_n^{(k)}\| = \lim\limits_{k\to\infty}\lim\limits_{n\to\infty}(\|x_n^{(k)}\| + \|y_n^{(k)}\|)$.

(ii) For every $\alpha > 0$, the following holds:
$\lim\limits_{k\to\infty}\liminf\limits_{n\to\infty}\|x_n^{(k)} + \alpha y_n^{(k)}\| = \lim\limits_{k\to\infty}\lim\limits_{n\to\infty}(\|x_n^{(k)}\| + \alpha\|y_n^{(k)}\|)$.

(iii) For some $\alpha > 0$, the following holds:

$$\lim_{k\to\infty}\liminf_{n\to\infty}\|x_n^{(k)} + \alpha y_n^{(k)}\| = \lim_{k\to\infty}\lim_{n\to\infty}(\|x_n^{(k)}\| + \alpha\|y_n^{(k)}\|).$$

In [9], Property M was introduced to be a necessary and sufficicient condition of that $K(X)$, the Banach space of all linear compact operator of a Banach space X, is M-ideal in $L(X)$, the Banach space of all continuous linear operator.

X has property M if $\liminf \|x_n - x\| = \liminf \|x_n - y\|$ for every weakly null sequence $\{x_n\}$ and $x, y \in X$ with $\|x\| = \|y\|$([9]). In Lemma 2.1 of [9], he showed that X has property M if and only if $\liminf \|x_n - x\| \leq \liminf \|x_n - y\|$ for every weakly null sequence $\{x_n\}$ and $x, y \in X$ with $\|x\| \leq \|y\|$.

We gives another characterization of Propety M by using a norm of ψ-direct sum $X \oplus_\psi X$.

Theorem 3. Let X be a Banach space. The following are equivalent.

(1)X has Property M;

(2)For every weakly null sequence $\{x_n\}$ of B_X, there exists $\psi \in \Psi$ such that $\liminf_n \|x_n - x\| = \|(\liminf_n \|x_n\|, \|x\|)\|_\psi$ for every $x \in B_X$.

By Theorem 3, we can prove the following propostion included in [5] without proof.

Propostion 4.([5]) Let X be a Banach space with Property M and $\{x_n\}$ a sequence converging weakly to x. Then

$$\liminf_n \|x_n\| \leq \liminf_m \liminf_n \|x_n - x_m\| + (\|x\| \vee \liminf_n \|x_n - x\| - \liminf_n \|x_n - x\|).$$

We recall that

$$R(1,X) = \sup\{\liminf \|x_n - x\| : x \in B_X, x_n \in B_x, x_n \text{ converges weakly } 0, \\ \liminf_m \liminf_n \|x_n - x_m\| \leq 1\}$$

Theorem 4. *Let X be a Banach space. If X has property M, then $R(1,X) \leq \frac{3}{2}$, i.e. $M(X) \geq \frac{4}{3} > 1$.*

We shall prove the following theorem by Propostion 2 , Lemma 1 and Lemma 2.

Theorem 5. Let $\psi \neq \psi_1$. $M(X \oplus_\psi Y) > 1$ if and only if $M(X) > 1$ and $M(Y) > 1$

By Theorem 5 and Theorem F, we obtain weak fixed point property for $X \oplus_\psi Y$.

Theorem 6. Let $\psi \neq \psi_1$. If $M(X) > 1$ and $M(Y) > 1$,then $X \oplus_\psi Y$ has weak fixed point property.

Theorem 3 and Theorem 4 provide the following corollary.

Corollary 1.([5]) *Let X and Y be Banach spaces and $\psi \in \Psi$. If X and Y have Property M and $\psi \neq \psi_1$, then $X \oplus_\psi Y$ has weak fixed point property for nonexpnasive mappings.*

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